11. ORDINAL NUMBERS

§11.1. Transitive Sets

We now turn our attention to ordinal numbers. With finite numbers the cardinal numbers, apart from zero, are 1, 2, 3, … while the ordinal numbers are 1^{st}, 2^{nd}, 3^{rd}, … There is really very little difference. For infinite sets there's a big difference. While cardinal numbers simply measure the size of a set, ordinal numbers describe the structure of a well-ordered set.

Consider the well-ordered set $\{1, 2, 3, \ldots, \aleph_0\}$. As a set, this is no bigger than $\{1, 2, 3, \ldots\}$. But as well-ordered sets the ordering is quite different. One set has a largest while the other does not.

A set x is **transitive** if $\cup x \subseteq x$, that is, if every element of an element of x is itself an element of x.

Example 1: Recall that $(1, 2) = \{\{1\}, \{1, 2\}\}$.
$x = \{0, 1, 2, \{2\}, \{1, 2\}, (1, 2)\}$ is not transitive since
$\cup x = \{0, 1, 2, \{1\}, \{1, 2\}\}$ and
$\{1\}$ is not an element of x.
However $y = \{0, 1, 2, \{2\}, \{1, 2\}, (2, 1)\}$ is transitive since
$y = \{\varnothing, \{0\}, \{0, 1\}, \{2\}, \{0\}, \{0, 1\}, \{\{2\}, \{2, 1\}\}\}$
$\quad = \{\varnothing, \{0\}, \{0, 1\}, \{2\}, \{1, 2\}, \{\{2\}, \{1, 2\}\}$
and so $\cup y = \{0, 0, 1, 2, 1, 2, \{2\}, \{1, 2\}\}$
$\quad = \{0, 1, 2, \{2\}, \{1, 2\}\}$ which is a subset of x.

The natural numbers are transitive, and the set of natural numbers itself is transitive.

Theorem 1: A set S is transitive if and only if $S \subseteq \wp(S)$.
Proof: Suppose S is transitive.
We must show that $S \subseteq \wp(S)$, that is, every element of S is a subset of S.
Let $y \in S$. If $z \in y$ then $z \in \cup S$ and so $z \in S$.
Hence $y \subseteq S$, as required.

Now suppose that $S \subseteq \wp(S)$. Let $y \in \cup S$.
Then for some z, $y \in z$ and $z \in S$.
Since $S \subseteq \wp(S)$, $z \in \wp(S)$, that is, $z \subseteq S$.
Since $y \in z$, then $y \in S$.
We have therefore shown that $\cup S \subseteq S$. ✋☺

Example 1 (continued): Observe that every element of y is a subset of y.

Theorem 2: If the elements of S are transitive then so are $\cap S$ and $\cup S$.
Proof: Suppose that the elements of S are transitive.
(1) Let $y \in \cup\cap S$.
We must show that $y \in \cap S$, that is $y \in u$ for all $u \in S$.
Let $u \in S$. Then by our assumption, u is transitive.
Since $y \in \cup\cap S$, $y \in z$ for some $z \in \cap S$.
Hence $z \in u$. Since $y \in z \in u$, $y \in u$.

(2) Let $y \in \cup\cup S$. Then for some z, $y \in z \in \cup S$.
Then $z \in u$ for some $u \in S$.
Since $y \in z \in u$ and u is transitive, $y \in u$.
Thus $y \in \cup S$. Hence $\cup x$ is transitive. ✋☺

Example 2: The sets $x = \{\{1, \{1\}\}, \{1\}, 0, 1, 2\}$ and $y = \{\{2, \{1\}\}, \{1\}, 0, 1, 2\}$ are transitive. Let $u = \{x, y\}$.
Then $\cap u = x \cap y = \{\{1\}, 0, 1, 2\}$ and
$\cup x = \{\{1, \{1\}\}, \{2, \{1\}\}, \{1\}, 0, 1, 2\}$, both of which are transitive.

§11.2. Ordinal Numbers

An **ordinal number** is a transitive set that is well-ordered by the relation '$\in$ or $=$'. So, for ordinals, $<$ and $\in$ are equivalent. We denote the class of ordinals by **Ord**.

Note that all natural numbers are ordinal numbers. So is the set $\mathbb{N}$ itself. It's odd to think of $\mathbb{N}$ as being a number. In fact, when we are thinking of $\mathbb{N}$ as a number we use a different symbol, ω. But remember that $\mathbb{N} = \omega$. Before we've finished we'll be using yet another symbol for $\mathbb{N}$.

If $(X, \leq)$ is a well-ordered set and is similar to the ordinal α we say that α is its **ordinal number**. We shall show that this is uniquely defined. We denote the ordinal of X by **ord (X, ≤)**. If the ordering is understood we can write **ord(X)**.

Theorem 3: If α is an ordinal number, then so is α^+. ✋☺

Theorem 4: Elements of ordinals are ordinals.
Proof: Elements of ordinals are subsets and so are well-ordered.
Let $x \in y \in z \in \alpha$, where α is an ordinal.
Then since α is transitive, x, y, z are elements of α.
Since $\in$ is transitive on α, $x \in z$. ✋☺

Theorem 5: Similar ordinals are equal.
Proof: Let $F:\alpha\rightarrow\beta$ be a similarity.
Let x be the smallest element of α such that $F(x) \neq x$.
If $y < x$ then $y = F(y) < F(x)$, whence $x \subseteq F(x)$.
Now suppose that $y < F(x)$.
Then, since F^{-1} is a similarity, $F^{-1}(y) < x$.
Hence $y = F(F^{-1}(y)) = FF^{-1}(y) < x$.
So $F(x) \subseteq x$, a contradiction. Hence there's no such x. ✋☺

Theorem 6: If α and β are ordinal numbers and $\alpha \subseteq \beta$ then $\alpha \in \beta$.
Proof: Suppose α, β are ordinals such that $\alpha \subset \beta$.
Let x be the least element of $\beta - \alpha$.
Since $x \in \beta$ and β is transitive, $x \subset \beta$.
Since $(\beta - \alpha) \cap x = 0$, $x \subseteq \alpha$.
Let $y \in \alpha$. Then $y \in \beta$.

Since $\{x, y\}$ has a least, either $x \in y$ or $x = y$ or $y \in x$.
If $x \in y$ or $x = y$, then $x \in \alpha$, contradicting the fact that

$$x \in \beta - \alpha.$$

Thus $y \in x$ and so $\alpha \subseteq x$.
But $x \subseteq \alpha$ and so $x = \alpha$. As $x \in \beta$, $\alpha \in \beta$. ✋☺

Theorem 7: Transitive subsets of ordinals are ordinals.
Proof: Suppose there is an ordinal α having a transitive subset which is not itself an ordinal, and suppose β is the least element of α^+ which has such a subset. Then there is some $X \subseteq \beta$ which is transitive but not an ordinal.

Hence X is not well-ordered by $\in$. Let $0 \neq Y \subseteq X$ and suppose that it has no least.

Now the elements of Y are elements of the ordinal β and so are ordinals, and hence transitive. Hence $\cap Y$ is transitive.

Let $y \in Y$. Then $\cap Y \subseteq y \in Y \subseteq X \subseteq \beta$. Hence $y \in \beta$. Thus y is an ordinal, and being less than β, the transitive subset $\cap Y$ must be an ordinal.
By Theorem 6, $\cap Y \in Y$. But clearly $\cap Y$ is the least element of Y, a contradiction. ✋☺

Theorem 8: If X is a set of ordinals then $\cup X$ is an ordinal.
Proof: Suppose X is a set of ordinals. Then $\cup X$ is transitive.
Let $0 \neq Y \subseteq \cup X$. Then the elements of Y are ordinals and so $\cap Y$ is transitive.

Let $\alpha \in Y$. Then $\cap Y \subseteq \alpha$. Thus $\cap Y$ is an ordinal.
Hence $\cap Y \in \alpha$ or $\cap Y = \alpha \in Y$.
If $\cap Y \notin Y$, $\cap Y \in \cap Y$, a contradiction.
Thus $\cap Y$ is the least element of Y.
Hence $\cup X$ is well-ordered by $\in$ and so $\cup X$ is an ordinal.
✋☺

The following theorem is known as a 'paradox' but it's just an ordinary theorem. It gets its name because a paradox arises if we claim that the class of ordinals is a set instead of a proper class.

Theorem 9 (BURALI-FORTI PARADOX): The class of ordinals is a proper class.
Proof: Suppose Ord is a set.
Then $\cup \text{Ord} \in \text{Ord}$ and so $\cup \text{Ord} \in (\cup \text{Ord})^+ \in \text{Ord}$.
Hence $\cup \text{Ord} \in \cup \text{Ord}$, a contradiction. ✋☺

§11.3. Transfinite Induction

The method of Proof by Induction is very useful in mathematics. It works because every non-empty set of natural numbers has a least, that is, the set of natural numbers is well-ordered by the usual ordering. Finite induction can be extended to infinite sets, provided we can well-order them.

Theorem 10 (PROOF BY TRANSFINITE INDUCTION):
Suppose W is a well-ordered set and $X \subseteq W$.
Suppose that X has the property that whenever all the predecessors of X are in X then so is X. Then X = W.
Proof: Suppose $X \subset W$. Then $m = \min(W - X)$, a contradiction! ✋☺

We can also define things by **transfinite induction**. The following is a special case.

Theorem 11: Suppose G is a generalized function whose domain is S, a subset of Ord with no maximum. Then there exists a unique function F on S such that $F(x^+) = G(F(x))$ for all $x \in S$ and $F(x) = \cup\{F(y) \mid y < x\}$ if x has no predecessor. ✋

§11.4. Ordinal Arithmetic

A non-zero ordinal is a **limit ordinal** if it has no immediate predecessor. An obvious example is ω, which is our alternative symbol for the set of natural numbers.

We define **addition** of ordinals by transfinite induction as follows:
(A0) $\alpha + 0 = \alpha$;
(A1) $\alpha + \beta^+ = (\alpha + \beta)^+$;
(A2) $\alpha + \beta = \cup \{\alpha + \gamma \mid \gamma < \beta\}$ if β is a limit ordinal.

Example 3: Successively adding 1 to ω we get the sequence $\omega + 1$, $\omega + 2$, …

We define **multiplication** of ordinals by transfinite induction as follows:

(M0) $\alpha 0 = 0$;

(M1) $\alpha\beta^+ = (\alpha\beta) + \alpha$;

(M2) $\alpha\beta = \cup \{\alpha\gamma \mid \gamma < \beta\}$ if β is a limit ordinal.

We define **exponentiation** of ordinals by transfinite induction as follows:

(E0) $\alpha^0 = 1$;

(E1) $\alpha^{\beta^+} = \alpha^\beta.\alpha$;

(E2) $\alpha^\beta = \cup\{\alpha + \gamma \mid \gamma < \beta\}$ if β is a limit ordinal and $\alpha \neq 0$;

(E3) $0^\beta = 0$ if β is a limit ordinal.

Examples 4:

$\omega + 1 = \omega + 0^+$
$= (\omega + 0)^+$ by (A1)
$= \omega^+$ by (A0)
$\neq \omega$.

$1 + \omega = \cup\{1 + n \mid n < \omega\}$
$= \omega$.

$2\omega = \cup\{2n \mid n < \omega\}$ by (M2)
$= \omega$.

$\omega 2 = \omega 1^+$
$= \omega 1 + \omega$ by (M1)
$= \omega 0^+ + \omega$

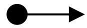
$= (\omega 0 + \omega) + \omega$ by (M1)
$= (0 + \omega) + \omega$ by (M0)
$= \cup\{0 + \mathrm{n} \mid n < \omega\} + \omega$
$= \omega + \omega$
$= \cup\ \{\omega + n \mid n < \omega\}$ by (A2).
$2^{\omega} = \cup\{2^{n} \mid n < \omega\}$ by (E2)
$= \omega$

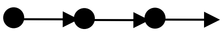
§11.5. Pictorial Representation of Ordinal Numbers

It's useful to be able to draw pictures of ordinal numbers, or at least some of them. We begin by representing a finite ordinal, *n*, by a row of *n* dots.

Example 5: The number 5 is represented by:

● ● ● ● ●

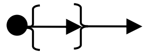
We could then represent ω by ● ● ● ····· but a more compact notation would be to have just one dot and to represent the row of small dots by an arrow: 
We can represent a + b by depicting a and then attaching a picture of b on its right.

Example 6: $\omega + 5$ would become 
$\omega 2$ would then be 
$\omega 2 + 3$ would be 
$\omega 3$ would be 
ω^{2} would be represented by an infinite row of the symbol for ω, or more simply by 

Each arrow is assumed to replicate whatever is before it an infinite number of times.

So $\omega^3 + \omega + 2$ would be 

Then what about ω^ω? This should be a dot followed by infinitely many arrows. We can depict this by bracketing an arrow and following this by an arrow:



In order that our convention that an arrow replicates everything on its left has to be modified to say that it replicates everything on its left back to the beginning or the next bracket.

To consider ordinals much beyond this would stretch our notation and also our ability to meaningfully picture them. However the ordinals go far beyond the level to which we can easily describe them.

§11.6. Definition of Cardinal Numbers

At long last we can define a cardinal number as a set. A **cardinal number** is simply an ordinal number that is not equivalent to any of its predecessors. This means that all of the natural numbers are also cardinal numbers. So too is the ordinal number ω, that we once knew as $\mathbb{N}$.

In honour of this new role for ω we use another notation, namely $\aleph_0$. So $\mathbb{N} = \omega = \aleph_0$. When we're thinking of it as a set we usually use the symbol $\mathbb{N}$. When we think of it as an ordinal number we write it as ω and when we think of it as a cardinal number we write it as $\aleph_0$.

SET		ORDINAL NUMBER		CARDINAL NUMBER
$\mathbb{N}$	=	ω	=	$\aleph_0$

Theorem 16: The class of ordinals that are equivalent to a set S is itself a set.

Proof: Well order $\wp(S)$ and let γ be the corresponding ordinal. Let Ord(S) be the class of ordinals that are equivalent to S.

Let α be an ordinal that is equivalent to S.

Then $\alpha < \gamma$, that is, $\alpha \in \gamma$.

So $\mathrm{Ord}(S) = \{\alpha \in \gamma \mid \alpha \approx S\}$. ✋☺

Now that we have established that Ord(S) is a set we can define the **cardinal number** of S to be **#S**, the smallest element of Ord(S).

We are now in a position to properly define the alephs, that is, to write every infinite cardinal number as $\aleph_\beta$ for some ordinal β.

Let γ be an infinite cardinal number. Let S_γ be the set of infinite cardinal numbers that are less than γ (less than in the sense of cardinal numbers). S_γ is well-ordered by $\leq$.

Let β be the ordinal number of this well-ordered set.

Then we denote γ by $\aleph_\beta$.

Example 7: Let $\beta = \aleph_\omega$. Then $S_\beta = \{\aleph_0, \aleph_1, \aleph_2, \ldots\}$ with $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$

The ordinal number of this well-ordered set is ω, which justifies the use of the notation $\aleph_\omega$.

Example 8: Let $\beta = \aleph_{\omega+1}$.

Then $S_\beta = \{\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega\}$ with

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_\omega$$

The ordinal number of this well-ordered set is $\omega + 1$, which justifies the use of the notation $\aleph_{\omega+1}$.

Theorem 17: If α is an ordinal number and $\#\alpha = \aleph_\beta$ then, as ordinals, $\aleph_\beta \leq \alpha < \aleph_{\beta+1}$.

Proof: $\aleph_\beta \leq \alpha$ by definition of cardinals.

If $\aleph_{\beta+1} \leq \alpha$ then $\aleph_{\beta+1} \leq \aleph_\beta$, a contradiction. ✋☺

Theorem 18: $\gamma = \cup\{\aleph_\alpha \mid \alpha < \beta\}$ is a cardinal number.

Proof: γ is an ordinal number.

Suppose $\aleph_\delta = \#\gamma$ and $\aleph_\delta < \gamma$.

Then $\aleph_\delta \in \gamma$, so $\aleph_\delta \in \aleph_\alpha$ for some $\alpha < \beta$.

Then $\aleph_\delta < \aleph_\alpha \leq \gamma$, a contradiction. ✋☺

Theorem 19: $\cup\{\aleph_\alpha \mid \alpha < \beta^+\} = \aleph_\beta$.

Proof: If $\alpha < \beta^+$ then $\alpha \leq \beta$ and so $\aleph_\alpha \subseteq \aleph_\beta$. ✋☺

Theorem 20: If β has no predecessor then

$$\cup\{\aleph_\alpha \mid \alpha < \beta\} = \aleph_\beta.$$

Proof: Let $\aleph_\gamma = \cup\{\aleph_\alpha \mid \alpha < \beta\}$.

For all $\alpha < \beta$, $\aleph_\alpha \subseteq \aleph_\beta$ and so $\aleph_\gamma \subseteq \aleph_\beta$.

Hence $\gamma \leq \beta$.

If $\gamma < \beta$, $\gamma + 1 < \beta$ so $\aleph_{\gamma+1} \subseteq \aleph_\gamma$, a contradiction. ✋☺